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DEFORMATION INSTABILITIES IN A THIN PLASTIC FILM ON AN ELASTIC SUBSTRATE DUE TO STRAIN MISMATCH

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Abstract—A model is constructed of a thin film of a rigid-plastic material bonded to a linear elastic substrate subject to a state of mismatch strain between the two materials. The purpose of the study is to investigate the existence of deformation localization in the film that is periodic along the direction of the bi-material interface. Bifurcation phenomena could be a possible precursor to debonding phenomena. The conditions necessary for these undulations to appear are given by a bifurcation equation which relates the internal film stress to the relative stiffness of the substrate and the thickness of the film.

1. INTRODUCTION

The study of thin film structures from the point of view of solid mechanics is becoming increasingly relevant to various areas of modern technology. Ultra-thin to moderately thick metallic and organic films are grown on relatively massive solid substrates for packaging purposes as well as for the actual manufacturing of microelectronic devices. The simplest structures consist of single films grown over relatively massive solid substrates and more complicated ones involve multi-layers of several films with different chemical, electrical, optical, thermal and mechanical properties. There are various methods used to grow films, but chemical vapor deposition and electron beam deposition are the two most commonly used techniques. Inherent to these deposition methods is the introduction of residual and thermal stresses in the thin films. Subsequent heat treatments can relieve some of these stresses, but other processes such as ion-implantation of the film–substrate composite can increase the internal film stress (see for instance Eernisse, 1977). Regardless of their origin, internal stresses can have deleterious effects on the optical, electrical and mechanical properties of thin films. It is therefore natural to be concerned with phenomena which lead to the degradation of the properties and, consequently, the quality and integrity of thin film structures. For instance, residual stresses can give rise to anomalous behavior in microelectronic devices that are made of such film–substrate composites. Ramirez *et al.* (1988) have recently studied the impact of residual stresses via the piezoelectric effect on some electrical properties of gallium arsenide field effect transistors.

Studies of thin film stability found in the literature deal primarily with thermal stability. Within this context, a thin film is said to be stable if it retains its desirable properties over a wide range of temperatures. The focus of this paper is on a different kind of stability, namely, mechanical stability. We are interested in the stability of a uniform state of deformation of a thin rigid-plastic film bonded to an elastic substrate in the presence of mismatch strain. Two important sources of this strain are lattice mismatch between the film and the substrate, and different thermal expansion coefficients. Within the framework of continuum mechanics, we focus on the existence of deformation instabilities that are periodic along the direction of the bi-material interface. These deformation modes can occur when the film is deformed in tension parallel to the interface and are not desirable in applications.

Various investigators, such as Weissmantel *et al.* (1979), have reported that high intrinsic stresses build up in thin film layers. A recent study on the shear stress distribution at a film–substrate interface is given by Freund and Hu (1988). As a result of these stress build-ups, films that exceed a certain thickness, or presumably a given internal stress level, tend to wrinkle and debond from the substrates. The process of wrinkling and debonding gives rise to observable patterns known as stress relief forms. Some of the patterns have been documented for diamond-like carbon thin films by Weissmantel *et al.* (1979), and more recently, by Nir (1984). It should be pointed out that these two studies dealt with hard films, and elastic buckling theories seem to be sufficient to adequately describe the observed phenomena. In the current study, the motivation for concentrating on deformation instabilities of plastic origin stems from the conjecture that the bulk deformation required to relax stresses, associated with differential thermal strains and lattice mismatch, is in many instances plastic flow (see Jackson and Li, 1982). Apparently, however, no systematic experimental study has been reported which deals precisely with the type of plastic instability which we analyze in this paper.

A solid body is said to be in a state of homogeneous deformation when it is subjected to a uniform state of strain. The departure from such a state and subsequent appearance of an inhomogeneous or localized deformation mode is known as a bifurcation phenomenon. The theoretical framework for this kind of phenomenon in plastic solids was pioneered by Biot (1965). Since then, much analytical and numerical work has been done in what has now become a very active research area. Most relevant to this study is the notable work of Hill and Hutchinson (1975). They considered bifurcation phenomena in the plane strain tension test of a block of material with a linearly incremental constitutive law. The present study, though not as exhaustive as theirs, follows the same methodology.

Also relevant to this work is a paper by Steif (1986) which examines interface instabilities in a bi-material composite consisting of a finite thickness layer between two infinite half spaces. In his work, Steif takes the same linearly incremental constitutive law for the materials composing the layer and the adjacent half spaces. In our study, only the layer is made of a linearly incremental material, whereas the substrate is taken to be linearly elastic. Less recently, Doris and Nemat-Nasser (1980) considered the problem of a layer on the surface of a half space subject to overall compressive loading (i.e. both layer *and* substrate experience the same compressive loading). These authors studied various combinations of constitutive laws for the layer and the half space. Even though the problem that they considered seems at first glance very similar to ours, our study differs in a fundmental way from theirs in that the loading of the bodies is due to an initial strain mismatch. This gives rise to an overall (compressive) loading of the layer and a compressive (tensile) one of the substrate. Moreover, we capitalize on the assumption that the layer is very thin in order to arrive at a very simple criterion for determining the onset of bifurcation.

In Section 2, the incremental boundary value problem to be solved in this paper is developed. Emphasis is placed on the assumptions made and on the general point of view that underlies this work. Section 3 begins with a summary of the constitutive equations that are used in the ensuing analysis. The section ends with the derivation of the velocity field that satisfies the kinematic constraints of the boundary value problem. In Section 4, the differential equations for the nominal stress rates are found. The bifurcation equation is determined in Section 5 by focusing on only one of the three governing equations available from Section 6. Finally, a discussion of the results is given in Section 7.

2. STATEMENT OF THE PROBLEM

We are interested in modelling a thin rigid-plastic film bonded to a linear elastic substrate of relatively large extent subject to a state of mismatch strain between the two materials. We assume that the substrate remains elastic while the film undergoes plastic deformation; this is a likely scenario when the elastic moduli of the film and the substrate differ by at least an order of magnitude, but the strengths are comparable. The geometry to be analyzed is depicted in Fig. 1. The thin film is centered about a local Cartesian coordinate system (x_1, x_2). The deformation is plane strain and is confined to the (x_1, x_2) plane. The film is assumed to be initially homogeneous and orthotropic (possibly isotropic) with respect to the x_1 and x_2 axes. Initially the substrate and the film are stress free. The film is then subjected to uniaxial homogeneous straining along the x_1-direction, with principal stretches along the x_1 and x_2 axes. At the instant under consideration (i.e. the pre-bifurcation state), the film has a thickness $2a_2$ and principal stresses $\sigma_1 = \sigma > 0$ and $\sigma_2 = 0$. We now inquire whether the film can wrinkle.

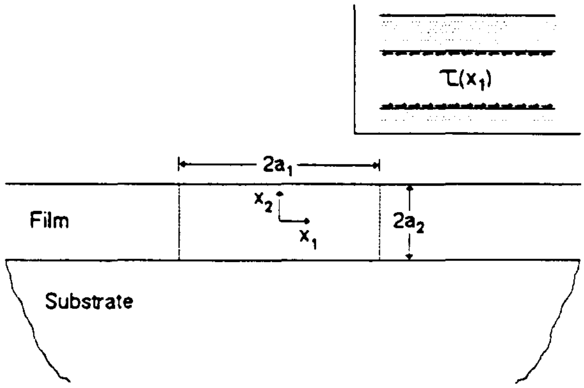

Fig. 1. Idealized geometry of the film–substrate composite and schematic representation of interfacial shear stress.

The point of view we adopt is that the state of uniaxial stress in the film is incurred during the fabrication of the composite, and we do not concern ourselves with the precise nature of the fabrication process. It should be mentioned that, for the case of strained layer epitaxy, the atomistic picture we have in mind is that the crystal in the substrate has larger (smaller) lattice dimensions than those of the material in the film. When the film is grown epitaxially on the substrate, tensile (compressive) stresses are introduced into the film due to the lattice mismatch.

A solution to the problem described above is obtained by choosing a linearly incremental constitutive law for the film, and by posing the boundary value problem in terms of rates of change of nominal stress $\dot{P}_{ij}$ and velocities v_i. For the remainder of this paper, roman subscripts will denote components of the tensors with respect to the x_1 and x_2 axis. In the ensuing analysis, periodic deformation modes are sought. This makes the boundary value problem tractable since we can exploit symmetry conditions by confining attention to a finite segment of the film, the length of which is equal to half the wavelength of the periodic surface deformation mode. As shown in Fig. 1, the current half-wavelength is taken to be equal to $2a_1$. Along the lateral edges of the segment, symmetry requires that the shear stress and the longitudinal component of velocity v_1 vanish. The top surface of the segment must remain traction free, and continuity of tractions and displacements between the film and the elastic substrate must be respected along the bottom face.

In the spirit of the thin film hypothesis, we only require continuity of displacements in the x_1-direction and of shear tractions along the film–substrate interface. Both requirements can be fulfilled simultaneously by requiring the extensional strain in the x_1-direction to be continuous across the interface at $x_2 = -a_2$. This is an equivalent statement of Hadamard's lemma in the context of the problem at hand. Since the proposed bifurcation analysis will be done by focusing on a finite segment of the rigid plastic film, a form of the continuity requirement is needed which involves only the film fields $\dot{P}_{ij}$ and v_i. In rate form, the aforementioned condition states that, at the film–substrate interface, the rate of change of the extensional strain $\dot{\varepsilon}^s_{11}$ of the substrate is equal to the spatial gradient in the x_1-direction of the longitudinal velocity component v_1 of the film. Our immediate goal is to find an expression for ε^s_{11} in terms of the interfacial shear stress $\tau(x_1)$ which is common to both film and substrate (see inset, Fig. 1). In order to extract the rate form of this expression, $\tau(x_1)$ will later be replaced by $\dot{P}_{21}(x_1, -a_2)$. In effect, we are stating that the Cauchy stress-rate and the nominal stress-rate are equal at the film–substrate interface, to insure consistency of the linear and non-linear theories along this boundary.

As previously stated, we assume that there is no traction acting along $x_2 = -a_2$ other than $\tau(x_1)$. Therefore, from plane strain elasticity, along the surface of the elastic half space ε^s_{11} is

$$\varepsilon^s_{11}(x_1, -a_2) = \frac{1-\nu^2}{E}\sigma_{11}(x_1, -a_2), \tag{2.1}$$

where $\sigma_{11}(x_1, -a_2)$ is the non-vanishing component of the Cauchy stress along the surface due to the tangential traction $\tau(x_1)$. Poisson's ratio and Young's modulus for the substrate are ν and E, respectively.

The Cauchy stress at the surface of the elastic half-space $x_2 = -2a_2$, due to a concentrated tangential load at $x_1 = \xi$ of magnitude $T(\xi)$, is obtained from the solution to the Flamant–Boussinesq problem (see Fung, 1965); $\sigma_{11}(x_1, -a_2) = -2T(\xi)/\pi(x_1-\xi)$. Replacing $T(\xi)$ by $\tau(\xi)\,d\xi$ and summing over the entire length of the film–substrate interface, results in the expression for $\sigma_{11}(x_1, -a_2)$ that we seek. Substituting this stress in (2.1) yields

$$\varepsilon^s_{11}(x_1, -a_2) = -\frac{2(1-\nu^2)}{\pi E}\int_{-t}^{t}\frac{\tau(\xi)}{x_1-\xi}\,d\xi. \tag{2.2}$$

We conclude this section by summarizing the boundary conditions for the incremental problem to be considered:

$$v_1(\pm a_1, x_2) = 0, \tag{2.3}$$

$$\dot{P}_{12}(\pm a_1, x_2) = 0, \tag{2.4}$$

$$\dot{P}_{22}(x_1, \pm a_2) = 0, \tag{2.5}$$

$$\dot{P}_{21}(x_1, a_2) = 0, \tag{2.6}$$

and

$$\frac{\partial v_1}{\partial x_1}(x_1, -a_2) = -\frac{2(1-\nu^2)}{\pi E}\int_{-t}^{t}\frac{\dot{P}_{21}(\xi, -a_2)}{x_1-\xi}\,d\xi. \tag{2.7}$$

3. CONSTITUTIVE EQUATIONS AND THE STREAM FUNCTION

For rate insensitive, incrementally linear solids in the rigid-plastic limit, the constitutive law is (see Hill and Hutchinson, 1975)

$$\begin{aligned}\overset{*}{\sigma}_{11}-\overset{*}{\sigma}_{22} &= 4\mu^*\dot{\varepsilon}_{11} = -4\mu^*\dot{\varepsilon}_{22},\\ \dot{\varepsilon}_{12} &= 0,\end{aligned} \tag{3.1}$$

where $\overset{*}{\sigma}_{ij}$ is the co-rotational rate of the Cauchy stress, and $\dot{\varepsilon}_{ij}$ is the spatial strain-rate. In an in-plane uniaxial test along either the x_1 or the x_2 axes, the tangent modulus is $4\mu^*$. Furthermore, in this constitutive law, hydrostatic pressure has no influence on the deformation.

In terms of nominal stress rates $\dot{P}_{ij}$ and velocities v_i, the constitutive relations (3.1) reduce to

$$\dot{P}_{11}-\dot{P}_{22} = \left(2\mu^*-\frac{\sigma}{2}\right)\left(\frac{\partial v_1}{\partial x_1}-\frac{\partial v_2}{\partial x_2}\right), \tag{3.2}$$

$$\dot{P}_{12}-\dot{P}_{21} = \sigma\frac{\partial v_2}{\partial x_1}, \tag{3.3}$$

where σ is the current principal Cauchy stress in the film along the x_1-axis. In addition to (3.2) and (3.3), the nominal stress rates must satisfy momentum balance. For incompressible materials, a useful form of the equations of incremental equilibrium is (see Hill and Hutchinson, 1975)

$$\frac{\partial}{\partial x_1}[\tfrac{1}{2}(\dot{P}_{11}-\dot{P}_{22})]+\frac{\partial}{\partial x_2}\dot{P}_{21} = -\frac{\partial}{\partial x_1}[\tfrac{1}{2}(\dot{P}_{11}+\dot{P}_{22})],$$

$$\frac{\partial}{\partial x_2}[\tfrac{1}{2}(\dot{P}_{11}-\dot{P}_{22})]-\frac{\partial}{\partial x_1}\dot{P}_{12} = \frac{\partial}{\partial x_2}[\tfrac{1}{2}(\dot{P}_{11}+\dot{P}_{22})]. \tag{3.4}$$

The incompressibility condition $\partial v_1/\partial x_1+\partial v_2/\partial x_2 = 0$ implies the existence of a stream function $\psi(x_1, x_2)$ such that

$$v_1 = \frac{\partial\psi}{\partial x_2}, \quad v_2 = -\frac{\partial\psi}{\partial x_1}. \tag{3.5}$$

It was shown in the original work of Biot (1965) or Hill and Hutchinson (1975) that in the rigid-plastic limit the governing equation for $\psi(x_1, x_2)$ is:

$$\frac{\partial^2\psi}{\partial x_1^2}-\frac{\partial^2\psi}{\partial x_2^2} = 0. \tag{3.6}$$

The presence of shear bands or slip lines in the film is closely related to the fact that the partial differential equation (3.6) is hyperbolic. Solutions of (3.6) give rise to families of pairs of characteristics in the (x_1, x_2) plane which bisect the coordinate directions (keep in mind that the coordinate directions are also the principal directions). The characteristic directions correspond to slip directions or lines along which shearing is localized.

As mentioned in Section 2, we search for deformation modes which are periodic in the x_1-direction. Such modes can be generated by choosing the following stream function:

$$\psi(x_1, x_2) = g(x_2)\cos(\eta_1 x_1). \tag{3.7}$$

In light of (3.6), the function $g(x_2)$ must satisfy

$$g''(x_2)+\eta_1^2 g(x_2) = 0. \tag{3.8}$$

The general solution to (3.8) can be written as

$$g(x_2) = A\sin(\eta_1 x_2+\phi), \tag{3.9}$$

where A is a constant amplitude, and ϕ is a phase angle. The key to successfully solving the current problem is not to consider modes which are symmetric or anti-symmetric about the x_1-axis, but rather take ϕ as an unknown. In this kind of analysis involving a homogeneous system of equations (see for instance Hill and Hutchinson, 1975), the amplitude A will remain undetermined and therefore can be set to unity at the outset. However, we choose to carry this constant along in the analysis until it is evident that it has no bearing on the final solution. For notational purposes, it is convenient to introduce the auxiliary functions $C(x_2) = A\cos(\eta_1 x_2+\phi)$ and $S(x_2) = A\sin(\eta_1 x_2+\phi)$.

In light of (3.9), the velocity field is obtained via (3.5) and may be written as

$$v_1(x_1, x_2) = \eta_1 C(x_2)\cos(\eta_1 x_1), \quad v_2(x_1, x_2) = \eta_1 S(x_2)\sin(\eta_1 x_1). \tag{3.10}$$

The kinematic constraint (2.3) is therefore satisfied when

$$\eta_1 = \frac{\pi n}{2a_1}, \quad n = 1, 3, 5, \cdots. \tag{3.11}$$

Moreover, it can be shown that the smoothness condition (2.4) is automatically satisfied by (3.10), and that the remaining boundary conditions are automatically satisfied by (3.8).

4. GOVERNING EQUATIONS FOR THE NOMINAL STRESS RATES

Substituting the velocities $v_1(x_1, x_2)$ and $v_2(x_1, x_2)$ as given in (3.10) into the constitutive relations (3.2) and (3.3) results in

$$\dot{P}_{11}-\dot{P}_{22} = (\sigma-4\mu^*)\eta_1^2 \sin(\eta_1 x_1)C(x_2), \tag{4.1}$$

$$\dot{P}_{12}-\dot{P}_{21} = \sigma\eta_1^2 \cos(\eta_1 x_1)S(x_2). \tag{4.2}$$

It should be noted that a feature of the rigid-plastic case is that there are no separate constitutive relations for $\dot{P}_{12}$ and $\dot{P}_{21}$. Consequently, the mathematical structure of these two stress rates may be obtained intuitively. The end condition (2.6) can be satisfied by taking

$$\dot{P}_{12}(x_1, x_2) = F(x_2)\cos(\eta_1 x_1), \tag{4.3}$$

where $F(x_2)$ is an unknown function. It follows from (4.2) that we must take

$$\dot{P}_{21}(x_1, x_2) = G(x_2)\cos(\eta_1 x_1), \tag{4.4}$$

where $G(x_2)$ is a second unknown function. Substituting (4.1), (4.3) and (4.4) into (3.4) yields

$$[\tfrac{1}{2}(\sigma-4\mu^*)\eta_1^3 C(\eta_1 x_2)+G'(x_2)]\cos(\eta_1 x_1) = -\frac{\partial}{\partial x_1}[\tfrac{1}{2}(\dot{P}_{11}+\dot{P}_{22})],$$

$$[\tfrac{1}{2}(4\mu^*-\sigma)\eta_1^3 S(\eta_1 x_2)+\eta_1 F(x_2)]\sin(\eta_1 x_1) = \frac{\partial}{\partial x_2}[\tfrac{1}{2}(\dot{P}_{11}+\dot{P}_{22})], \tag{4.5}$$

where prime denotes differentiation with respect to x_2. These last two differential equations suggest that we take

$$\tfrac{1}{2}[\dot{P}_{11}(x_1, x_2)+\dot{P}_{22}(x_1, x_2)] = H(x_2)\sin(\eta_1 x_1), \tag{4.6}$$

where $H(x_2)$ is the third and last unknown function that is introduced.

We can obtain governing equations in terms of the unknown functions $F(x_2)$, $G(x_2)$ and $H(x_2)$ by appropriately substituting the expressions (4.3), (4.4) and (4.6) into the equations (4.2) and (4.5). The resulting relations are

$$F(x_2)-G(x_2) = \sigma\eta_1^2 S(x_2), \tag{4.7}$$

$$G'(x_2)+\eta_1 H(x_2) = \tfrac{1}{2}(4\mu^*-\sigma)\eta_1^3 C(x_2), \tag{4.8}$$

$$H'(x_2)-\eta_1 F(x_2) = \tfrac{1}{2}(4\mu^*-\sigma)\eta_1^3 S(x_2). \tag{4.9}$$

Equations (4.7) through (4.9) can be suitably manipulated to yield uncoupled second order differential equations for $F(x_2)$, $G(x_2)$ and $H(x_2)$. We can obtain the sought after bifurcation equation from any one of the three resulting equations. We will proceed to derive only the governing equation for $G(x_2)$. Naturally, all three differential equations are needed if one wishes to solve explicitly for the nominal stress rates $\dot{P}_{ij}(x_1, x_2)$ via equations (4.1), (4.3), (4.4) and (4.6). These steps offer no added mathematical difficulty and will be omitted from the current discussion. Differentiating (4.8) and substituting the resulting expression for $H'(x_2)$ from (4.9) and for $F(x_2)$ from (4.7) yields the following governing equation for $G(x_2)$:

$$G''(x_2)+\eta_1^2 G(x_2) = -4\mu^*\eta_1^4 S(x_2). \tag{4.10}$$

The differential equation (4.10) must be solved subject to the boundary conditions

$$G(a_2) = 0, \tag{4.11}$$

$$G'(a_2) = (4\mu^*-\sigma)\eta_1^3 C(a_2), \tag{4.12}$$

$$G(-a_2) = \frac{E}{2(1-\nu^2)}\eta_1^2 C(-a_2), \tag{4.13}$$

$$G'(-a_2) = (4\mu^*-\sigma)\eta_1^3 C(-a_2). \tag{4.14}$$

There boundary conditions were obtained by expressing the original boundary conditions (2.5) through (2.7) in terms of the unknown functions $F(x_2)$, $G(x_2)$ and $H(x_2)$, and then making judicious use of equations (4.7) through (4.9).

5. BIFURCATION EQUATION

The general solution of the differential equation (4.10) is

$$G(x_2) = c_0 \sin(\eta_1 x_2) + c_1 \cos(\eta_1 x_2) + 2\mu^*\eta_1^3 x_2 C(x_2) - \mu^*\eta_1^2 S(x_2), \tag{5.1}$$

where c_0 and c_1 are integration constants that must be determined from the boundary conditions (4.11) through (4.14). Let $G_+(x_2)$ be the solution to (4.10) that satisfies the boundary conditions (4.11) and (4.12). Note that these boundary conditions are prescribed along the top face of the film. Likewise, let $G_-(x_2)$ be the solution which satisfies the boundary conditions (4.13) and (4.14). The solution to (4.10) that we seek must satisfy the boundary conditions at the top and bottom faces of the film. By requiring that $G_+(x_2)$ and $G_-(x_2)$ be equal to each other, we can obtain a constraint among σ, μ^*, a_1, a_2, η_1, E and ν. The resulting expression is an eigenvalue equation and is usually identified as the bifurcation equation. Either solution $G_+(x_2)$ or $G_-(x_2)$, when used in conjunction with the bifurcation equation, will satisfy both sets of boundary conditions. Alternatively, one can apply the boundary conditions (4.11)–(4.14) to the solution (5.1) and obtain the bifurcation condition from the requirement that the integration constants c_0 and c_1 cannot both be zero.

The individual expressions for $G_+(x_2)$ and $G_-(x_2)$ are rather long and unwieldy, and are of no use to us except for obtaining the eigenvalue equation. For this reason, we choose not to display them here but rather directly provide the simpler expression for $G_+(x_2)-G_-(x_2) = 0$,

$$\frac{A}{2}[\eta_1^2(\sigma-2\mu^*)\sin(\eta_1 x_2+2\eta_1 a_2-\phi)-\eta_1^2 E^*\cos(\eta_1 x_2+2\eta_1 a_2-\phi)$$
$$+\eta_1^2(2\mu^*-\sigma)\sin(\eta_1 x_2-2\eta_1 a_2-\phi)-\eta_1^2(8\eta_1 a_2\mu^*+E^*)\cos(\eta_1 x_2+\phi)] = 0, \tag{5.2}$$

where $E^* = E/2(1-\nu^2)$. One can expand the trigonometric functions in (5.2) and rewrite the resulting expression as the sum of a sin $(\eta_1 x_2)$ term and a cos $(\eta_1 x_2)$ term being equal to zero. In general, the latter equation can hold only if the coefficients of sin $(\eta_1 x_2)$ and cos $(\eta_1 x_2)$ identically vanish. This last requirement gives rise to the following two equations:

$$2(\sigma-2\mu^*) = -\frac{E^*\sin(2\eta_1 a_2-\phi)+(8\eta_1 a_2\mu^*+E^*)\sin\phi}{\sin(2\eta_1 a_2)\sin\phi}, \tag{5.3}$$

$$2(\sigma-2\mu^*) = \frac{E^*\cos(2\eta_1 a_2-\phi)+(8\eta_1 a_2\mu^*+E^*)\cos\phi}{\sin(2\eta_1 a_2)\cos\phi}. \tag{5.4}$$

We can solve for the phase angle ϕ by subtracting (5.4) from (5.3) and expanding the trigonometric functions. The result is

$$\phi = \tfrac{1}{2}\arcsin\left(\frac{-E^*\sin(2\eta_1 a_2)}{8\eta_1 a_2\mu^* + E^*}\right). \tag{5.5}$$

Furthermore, (5.4) can be rewritten as

$$\bar{\sigma} = \tfrac{1}{2} + q\csc(2q) + \frac{\xi}{8}[\cot q + \tan\phi], \tag{5.6}$$

where the non-dimensional variables $\bar{\sigma} = \sigma/4\mu^*$, $\xi = E^*/\mu^*$, and $q = \eta_1 a_2$ have been introduced. The variable q corresponds to the non-dimensional wave number; when $q \downarrow 0$ the wavelength of the inhomogeneous deformation modes becomes arbitrarily large with respect to the thickness of the film, and when $q \uparrow \infty$ the wavelength becomes arbitrarily small. Likewise, ξ is a measure of the relative stiffness of the substrate; when $\xi = 0$ the substrate is infinitely compliant, and as $\xi \uparrow \infty$ the substrate becomes perfectly rigid. The tangent of ϕ which appears in equation (5.6) is easily obtained from (5.5) via the trigonometric identity $\sin(2\phi) = 2\tan\phi/(1+\tan^2\phi)$ as

$$\tan\phi = \frac{8q+\xi}{\xi\sin(2q)}\left\{-1 \pm \sqrt{1-\left(\frac{\xi\sin(2q)}{8q+\xi}\right)^2}\right\}. \tag{5.7}$$

Equation (5.6) is the so-called bifurcation equation which must be satisfied if deformation instabilities are to be possible. In the absence of the elastic substrate, $\xi = 0$, this equation reduces to

$$\bar{\sigma} = \tfrac{1}{2} \pm \frac{q}{\sin(2q)}. \tag{5.8}$$

As expected, equation (5.8) is equivalent to equation (AI.11) in Hill and Hutchinson (1975) where the plus (minus) sign corresponds to symmetric (anti-symmetric) incremental deformation modes.

An inspection of (5.6) reveals that the eigenstress $\bar{\sigma}$ is singular when $q = n\pi/2$, $n = 1, 3, 5, \cdots$. Moreover, $\bar{\sigma}$ is also singular when $q = 0$. This is unlike the case of a rigid-plastic block in plane tension, for which the bifurcation equation is (5.8), where $\bar{\sigma} \downarrow 1$ as $q \downarrow 0$. The stress level $\sigma = 4\mu^*$, or $\bar{\sigma} = 1$, is the so-called maximum load. Finally, it should be pointed out that as the substrate becomes increasingly rigid the eigenstress required for bifurcation becomes unbounded, but it does so in a non-uniform fashion. This completes the determination of the bifurcation equation (5.6) for the boundary value problem that was posed in Section 2.

6. DISCUSSION OF RESULTS

We focus attention on the first order symmetric mode which is obtained by taking the plus sign in (5.7) and letting $n = 1$ in (3.11), that is, $q = \pi a_2/2a_1$. This is the eigenmode that has physical significance, for an examination of (5.5) will reveal that the least tensile eigenstress $\bar{\sigma}$ will occur when $a_2/a_1 < 1$. Likewise, if the film is under compression one would only have to consider the first order anti-symmetric mode. In this symmetric regime $\bar{\sigma}$ is unbounded when $q = 0$ and when $q = \pi/2$. This means that there is an absolute minimum of $\bar{\sigma}$, say $\bar{\sigma}_c$, when q is in the range $0 < q < \pi/2$. There is at least one value of the non-dimensional wave number, denoted by q_c, corresponding to $\bar{\sigma}_c$.

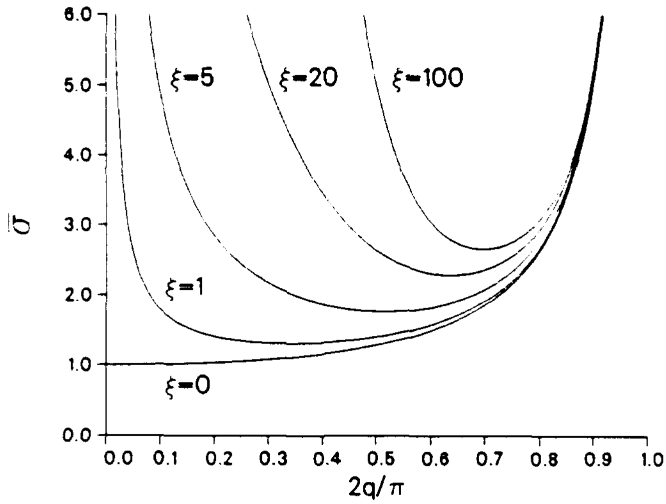

Fig. 2. Plot of the relation between the eigenstress and the non-dimensional wave number for several values of the relative stiffness of the substrate.

Figure 2 shows curves of $\bar{\sigma}$ versus q for several values of ξ. The curve for $\xi = 0$ is the result of Hill and Hutchinson (1975) for the bifurcation of a rigid-plastic block in plane tension. As previously discussed, for non-zero values of ξ there is a critical value q_c for which bifurcation from the homogeneous state is possible. A consequence of this is that when a film of thickness $2a_2$ reaches the appropriate minimum eigenstress $\bar{\sigma}$, the inhomogeneous periodic deformation modes that can be observed will have a wavelength $4a_1$ equal to $2\pi a_2/q_c$. In the same spirit, one can also state that if $\bar{\sigma}_c$ is not achieved in the film, then sinusoidal modes of the form (3.7) will not be observed. Other conclusions may be drawn from Fig. 2 by fixing the value of the eigenstress $\bar{\sigma}$ rather than fixing the thickness of the film.

The above observations are illustrated in Fig. 3 where the bifurcation curve is shown for the case of ξ equal to 0.25. This value of ξ is obtained by considering a substrate with the elastic properties of polyethylene, a film made of a power-law material with the properties of copper, and a mismatch strain of 2%. The Young's modulus and Poisson's ratio of polyethylene are assumed to be $2 \times 10^8 \mathrm{N/m^2}$ and 0.45, respectively. The stiffness parameter and the hardening rate for copper (99.94% pure) are taken to be 4.5×10^8 $\mathrm{N/m^2}$ and 0.33,

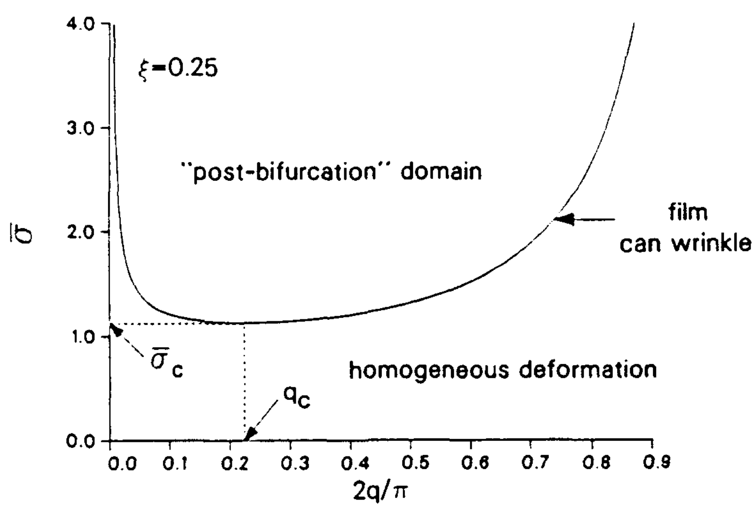

Fig. 3. Plot of the relation between the eigenstress and the non-dimensional wave number for the case of $\xi = 0.25$. The analytical model cannot predict the nature of the inhomogeneous deformation in the post-bifurcation domain, i.e. for values above $\bar{\sigma}_c$.

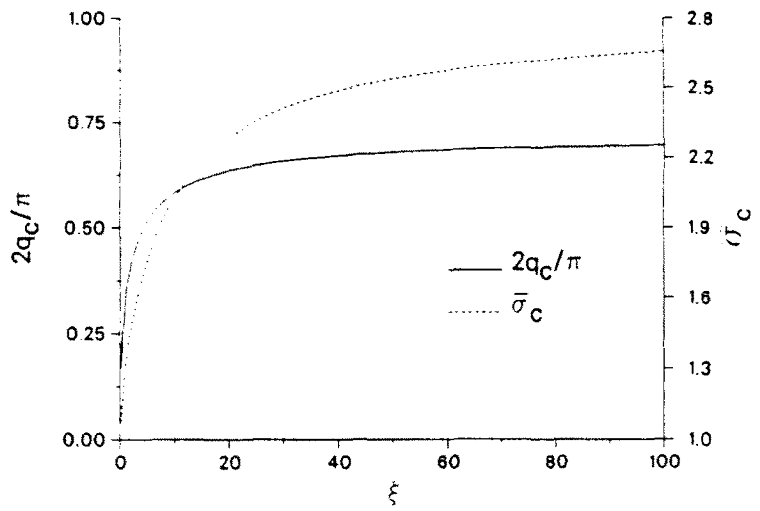

Fig. 4. Plots of the critical values of the eigenstress and the non-dimensional wave number as a function of the variable ξ.

respectively. The model predicts that the wavelength of the inhomogeneous deformation is almost nine times the thickness of the film. This wavelength mode is sufficiently long to be consistent with the thin film hypothesis introduced in Section 2.

One must keep in mind that when using values of ξ which result in short-wavelength modes, the thin film assumption, which requires that the various field quantities vary on length scales which are large compared with the thickness of the film, breaks down. In this case, one could relax the thin film idealization in the original statement of the boundary value problem and require continuity of transverse displacements and of normal tractions. Instead of $\dot{P}_{22}(x_1, -a_2) = 0$ in (2.5), one would have an expression relating $\dot{P}_{22}$ to v_2 along the interface. Then, restricting attention to small film thicknesses, one could see how different the predictions from the improved model are from those presented in this paper. For small values of ξ the differences are expected to be insignificant.

Another observation that can be made from Fig. 2 is that as the degree of constraint of the substrate on the film increases, higher eigenstresses are required to meet the bifurcation condition. This in turn means that, for a fixed film thickness, as the substrate becomes stiffer the wavelength of the inhomogeneous deformation mode becomes shorter. Figure 4 shows the locus of values of $\bar{\sigma}_c$ and q_c for a realistic range of the non-dimensional variable ξ. Over most of the range of ξ shown in the plot the internal film stress necessary for bifurcation is approximately two and a half times the maximum load stress.

7. CONCLUSIONS

In the previous sections the mechanical stability of a strained thin plastic film bonded to an elastic substrate has been analyzed. It is found that periodic surface instabilities can occur and that the appearance of these undulations depends on the amount of internal stress in the film, the rigidity of the substrate, and the thickness of the film. Unlike the simpler case of the plane tension test of a rigid-plastic block, where departure from homogeneous deformation can take place just following maximum load, bifurcation phenomena is only possible at stress levels higher than maximum load. Moreover, the analysis reveals that having fixed the film thickness and material parameters, bifurcation can only occur for a critical value of stress and a critical wavelength of the inhomogeneous deformation mode.

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REFERENCES

Biot, M. A. (1965). *Mechanics of Incremental Deformations*. Wiley, New York.

Dorris, J. F. and Nemat-Nasser, S. (1980). Instability of a layer on a half space. *J. appl. Mech.* **47**, 304–312.

Eernisse, E. P. (1977). Stress in ion-implanted CVD Si_3N_4 films. *J. appl. Phys.* **48**, 3337–3341.

Freund, L. B. and Hu, Y. (January 1988). Shear stress at a film–substrate interface due to mismatch strain. Brown University Report.

Fung, Y. C. (1965). *Foundations of Solid Mechanics*. Prentice–Hall, Englewood Cliffs, NJ.

Hill, R. and Hutchinson, J. W. (1975). Bifurcation phenomena in the plane tension test. *J. Mech. Phys. Solids* **23**, 239–264.

Jackson, M. S. and Li, C.-Y. (1982). Stress relaxation and hillock growth in thin films. *Acta Metal.* **30**, 1993–2000.

Nir, D. (1984). Stress relief forms of diamond-like carbon thin films under internal compressive stress. *Thin Solid Films* **112**, 41–49.

Ramirez, J.-C., McNally, P. J., Cooper, L. S., Rosenberg, J. J., Freund, L. B. and Jackson, T. N. (1988). Development and experimental verification of a two-dimensional numerical model of piezoelectrically induced threshold voltage shifts in GaAs MESFETs. *IEEE Trans. Electron Devices* **35**, 1232–1240.

Steif, P. S. (1986). Bimaterial interface instabilities in plastic solids. *Int. J. Solids Structures* **22**, 195–207.

Weissmantel, C., Reisse, G., Erler, H.-J., Henny, F., Bewilogua, K., Ebersbach, U. and Schurer, C. (1979). Preparation of hard coatings by ion beam methods. *Thin Solid Films* **63**, 315–325.